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Some Geometric Results in Semidefinite Programming

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Abstract. The purpose of this paper is to develop certain geometric results concerning the feasible regions of Semidefinite Programs, called here *Spectrahedra*.

We first develop a characterization for the faces of spectrahedra. More specifically, given a point x in a spectrahedron, we derive an expression for the minimal face containing x. Among other things, this is shown to yield characterizations for extreme points and extreme rays of spectrahedra. We then introduce the notion of an algebraic polar of a spectrahedron, and present its relation to the usual geometric polar.

Key words: Semidefinite programming, convex geometry.

1. Introduction and Motivation

Let $\mathcal{S}_n, \mathcal{P}_n$ denote respectively, the space of n-by-n real symmetric matrices and the cone of $n \times n$ positive semidefinite (PSD) matrices. We let $\succeq$ denote the Loewner partial order induced by $\mathcal{P}_n$ on $\mathcal{S}_n$, i.e., $A \succeq B$ if $A - B$ is positive semidefinite.

DEFINITION 1. A *Spectrahedron* is a closed convex set of the following type:

$$G = \{x | Q(x) \succeq 0\},$$

where

$$Q(x) = Q_0 + \sum_{i=1}^{m} x_i Q_i$$

with $Q_i \in \mathcal{S}_n \ \forall i = 0, \ldots, m$.

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Since the smallest-eigenvalue function is concave, it follows that G is a closed convex set. In this paper, we present some geometrical results concerning spectrahedra.

Spectrahedra are nothing but the feasible regions of Semidefinite Programs (SDP) [1, 2]. The name *Spectrahedron* can perhaps be justified as follows: the definition of this class of sets involves the spectrum, and they bear a resemblance to polyhedra. Indeed, spectrahedra may be considered "next natural successors" to polyhedra, as one moves beyond linear constraints in optimization theory.

1.1. BACKGROUND

Historically, semidefinite programming has been studied in more general contexts such as convex and cone programming (see [6], [8], [9] and [32]). See also [11] and [21]. Further references can be found in [2].

However, the more recent surge of interest in SDP was primarily inspired by the work of [14] (see [15], Chapter 9). In this work, the authors associate with every graph G, a convex set denoted by $\mathrm{TH}(G)$, and show that when G is perfect, this set equals the stable set polytope. Then they demonstrate that one can optimize over $\mathrm{TH}(G)$ in polynomial time, and hence the stable set problem (along with many other related problems) can be solved in polynomial time for perfect graphs. (In our language, the convex set $\mathrm{TH}(G)$ is the projection of certain spectrahedron onto a subspace.)

The algorithms of [14] employ the ellipsoid method, and are not considered to be efficient in practice. In his Ph.D. work, Farid Alizadeh showed that one can extend, in an almost mechanical fashion, many of the known interior point methods for LP into polynomial time algorithms for solving SDPs approximately. Independently of Alizadeh's work, Nesterov and Nemirovskii [20] developed efficient interior point methods for a wider class of convex programs, by employing self-concordant barrier functions. We refer the reader to [2] and [31] for an account of several algorithmic approaches to SDP as well as its applications.

A very recent result of Goemans and Williamson [12] showing that one can use the solution obtained from a semidefinite relaxation to obtain a 0.878-approximation algorithm[1] for the Max-Cut problem, gives further impetus to Semidefinite Programming. Their result employs an ingenious randomized rounding scheme. This result has inspired other recent results on the application of SDP to combinatorial optimization problems.

A complete duality theory has recently been developed in [26]. The resulting dual, called the Extended Lagrange–Slater Dual (ELSD), is an explicit polynomial size semidefinite program, which enjoys zero-duality gap, and yields several complexity results for semidefinite programming. The derivation of ELSD arose as an extension of the analysis of polars of spectrahedra developed in Section 3 (in particular, proof of Theorem 2) of the current paper.

In [18], Laurent and Poljak investigate certain geometric features of the set of correlation matrices, which form a special class of spectrahedra, defined by:

$$\tilde{\mathcal{L}}_n = \{X | X \in \mathcal{P}_n, X_{ii} = 1 \ \forall i\}.$$

This set, referred to as an *Elliptope* by the authors, is precisely the feasible region of the SDP employed by Goemans and Williamson. In [24] and [25], Pataki developed certain results for the faces of spectrahedra and applied them to solve SDPs in the framework of Semi-Infinite Linear Programming.

1.2. Relations to Multiquadratic Programming

The Multiquadratic Programming Problem (MQP) is the problem of minimizing a quadratic objective function subject to quadratic equality and inequality constraints. Our interest in spectrahedra and semidefinite programming was originally motivated by their relation to MQP. Essentially, SDP arises as a relaxation of MQP, which we explain as follows.

Consider the MQP $\min\{g(x)|f(x) = 0\}$, where $g(x) = x^T Q_0 x + b_0^T x$ and $f_i(x) = x^T Q_i(x) + b_i^T x + c_i, \forall i = 1, \ldots, m$. Now define $G(U, x) = U \cdot Q_0 + b_0^T x$ and $F_i(U, x) = U \cdot Q_i + B_i^T x + c_i, \forall i = 1, \ldots, m$, where U is a symmetric matrix variable. Clearly, the MQP is equivalent to $\min\{G(U, x)|F(U, x) = 0, U - xx^T = 0\}$. The MQP is NP-Hard, and hence it is natural to consider its relaxations. We consider, in particular, relaxing $U - xx^T = 0$ to $U - xx^T \succeq 0$, or equivalently,

$$\begin{bmatrix} U & x \\ x^T & 1 \end{bmatrix} \succeq 0.$$

The relaxation thus obtained is a semidefinite program, and is called the *Image Convexification Relaxation (ICR)* as one can show that [27, 29]

$$\mathrm{Conv}(f(\Re^n)) = \{F(U, x)|U - xx^T \succeq 0\}.$$

The ICR is closely related to the N_+ operator defined in [19]. In particular, the semidefinite relaxation of the stable set problem as considered by (see [15] and [19]) as well as that of the Max-Cut problem as in [12] are precisely the ICRs of the corresponding MQPs: $\max\{e^T x | x_i x_j = 0 \ \forall (i, j) \in E, x_i^2 = x_i \ \forall i\}$ in the case of the former, and $\max\{x^T Q x | x_i^2 = 1 \ \forall i\}$ for the latter.

It is NP-Hard to check whether the ICR of a given MQP is exact, i.e., whether the optimum value of the semidefinite relaxation is equal to that of the MQP; this was shown for the special case of Max-Cut problem in [10] and [18]. The problem of determining whether the ICR is exact for every c (constant term of $f(x)$) reduces essentially to checking whether the image of a quadratic map is convex. In [29], it was shown that this latter problem is also NP-Hard. However, the problem of checking if the image of every subspace under a given quadratic map is convex can be accomplished in polynomial time.

On the other hand, in order to solve the original MQP exactly, one needs to maintain $U - xx^T = 0$. One way of doing so is to require that there exist y such that

$$\begin{bmatrix} U & x \\ x^T & 1 \end{bmatrix} = yy^T.$$

Such optimization problems are called *Unary Programs* in [27] and [28], wherein, certain valid cuts for the feasible regions of Unary programs were developed. These cuts, similar in flavor to the Gomory/Chvátal cuts for Integer Programming, are based on some eigenvalue inequalities called the Weyl inequalities.

1.3. NOTATION

In large part, we will follow the matrix and convex theoretic notations of respectively, [16] and [30].

We denote by $\mathcal{M}_{m,n}$, $\mathcal{M}_n$ and $\mathcal{S}_n$, the spaces of $m \times n$ real matrices, $n \times n$ real matrices, and its subspace of symmetric matrices, respectively. For $U \in \mathcal{S}_n$, we write $U \succeq 0$ (resp. $\succ$), if U is positive semidefinite (resp. positive definite), i.e. all the eigenvalues of U are nonnegative (resp. positive). The inner product on $\mathcal{M}_{m,n}$ (and $\mathcal{S}_n$) is given by:

$$A \cdot B = \sum_{i,j} A_{ij} B_{ij}.$$

Given $A \in \mathcal{M}_{m,n}$, $B \in \mathcal{M}_{k,l}$, their *direct sum* is the block partitioned $(m + k) \times (n + l)$ matrix:

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

As usual, $\mathrm{Diag}(v)$ for $v \in \Re^n$ denotes the diagonal matrix formed from v. For any $R \in \mathcal{M}_{m,n}$, $\mathrm{Null}(R)$ denotes the null space of R.

The *spectral decomposition* of a symmetric matrix A is $A = V^T DV$, where V is an orthogonal matrix (i.e. $V^T V = I$) of eigenvectors of A, and D is the diagonal matrix of the eigenvalues of A, and the spectral radius, denoted by $\rho(A)$, is the largest of the magnitudes of the eigenvalues A. A collection $\{A_1, \ldots, A_k\}$ of matrices in $\mathcal{S}_n$ is said to be *simultaneously diagonalizable via congruence*, if there exists a nonsingular matrix S such that each of the $S^T A_i S$ is diagonal.

For $A, B \subset \Re^n$, $A + B$ denotes the *Minkowski Sum* (also called the *set sum*). For $A \subset \Re^n$, $\mathrm{Conv}(A)$ (resp. $\mathrm{Aff}(A)$) denotes the smallest convex set (resp. affine subspace) containing A. The dimension of A is $\dim(\mathrm{Aff}(A))$, and $B(x, r)$ is the ball of radius r around x.

Let $G \subset \Re^n$ be a convex set. The interior and the relative interior of G are denoted by $\mathrm{Int}(G)$ and $\mathrm{ri}(G)$ respectively. The *recession cone* of G is defined by:

$$0^+(G) = \{v \in \Re^n | \forall x \in G, t \geqslant 0, x + tv \in G\},$$

and its *lineal hull* (contrast with linear hull) is the subspace $0^+(G) \cap 0^+(-G)$.

A *face* of a convex set G is a convex subset F such that whenever $a, b \in G$, $(a, b) \cap F \neq \emptyset$, we have $a, b \in F$ (here, (a, b) denotes the open segment that "joins" a and b). Any convex set is the disjoint union of the relative interiors of its faces ([30], S 18). In other words, given any $z \in G$, there is a unique face which contains z in its relative interior. We call this face the *minimal face* of z, and denote it by $F_G(z)$. A face F of G is said to be *exposed*, if it is the intersection of some hyperplane with G or it is G itself.

The *polar* of a convex set G is

$$G^\circ = \{y | y^T x \leqslant 1 \;\; \forall x \in G\}.$$

A generic spectrahedron will be denoted by $G = \{x | Q(x) \succeq 0\}$, corresponding to a matrix map $Q(x)$. We will let $\hat{Q}(x)$ denote the linear part of $Q(x)$, i.e. $Q(x) - Q_0$, and $N(x)$ stand for the null space $\mathrm{Null}(Q(x))$. At certain places in this paper, we will consider $Q(x)$ in its *reduced form*, which is defined as follows. Let $V^T D' V$ be the spectral decomposition of Q_0, where $D' = D \oplus 0$ and $D_{ii} \neq 0 \, \forall i$. Then the reduced form is:

$$Q'(x) = V^T Q(x) V = \begin{bmatrix} D + A(x) & B(x)^T \\ B(x) & C(x) \end{bmatrix}.$$

Here, $A(x)$, $B(x)$, and $C(x)$ are linear matrix maps of appropriate sizes. In particular, when $0 \in G$, or equivalently $Q_0 \succeq 0$, then D is positive definite.

1.4. PRELIMINARIES

The following well known facts will be used in the paper (the proofs in most part can be found in [16] and [30]).

- If $U \succeq 0$, and $U_{ii} = 0$, then $U_{ij} = 0 \, \forall j$.
- If $A \in \mathcal{S}_n$ and $S \in \mathcal{M}_n$ and nonsingular, then $A \succeq 0$ iff $S^T A S \succeq 0$. (Special case of the Sylvester's law of inertia.)
- If $A \succeq 0$ and $u \in \Re^n$, then $Au = 0$ iff $u^T A u = 0$.
- Given a block partitioned matrix

  $$U = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix},$$

  where A and C are square, and A is nonsingular, then the *Schur Complement* of A in U is the matrix $S = C - BA^{-1}B^T$. We have that, if $A \succ 0$, then $U \succeq 0 \Leftrightarrow S \succeq 0$ and $U \succ 0 \Leftrightarrow S \succ 0$.
- $A \in \mathcal{S}_n$ is PSD iff $A \cdot B \geqslant 0$ for all $B \succeq 0$.
- If $A, B \succeq 0$, then $A \cdot B = 0$ iff $AB = 0$ (see [2] for a proof).
- If A, B are convex sets, and $A \subset B$, then $A^\circ \supset B^\circ$. And, if A is a closed convex set containing the origin, then $A^{\circ\circ} = A$.
- If G is a cone, then $G^\circ = \{y | y^T x \leqslant 0 \;\; \forall x \in G\}$.

- If K_i, $i = 1, \ldots, l$ are cones, then $(K_1 + K_2 + \ldots + K_l)^\circ = \cap_i K_i^\circ$.

Here are some easy observations that one can make about spectrahedra:

1. Every polyhedron is a spectrahedron: Given $P = \{x | Ax \geqslant b\}$, take $Q(x) = \mathrm{Diag}(Ax - b)$.
2. The intersection of two spectrahedra is another spectrahedron: Take the direct sum of the two maps involved.
3. The property of spectrahedrality is unaltered by bijective affine transformations of the space.
4. A spectrahedron remains unaltered when a nonsingular congruence transformation is applied to the matrix map: i.e. if V is a nonsingular matrix, then $G = \{x | V^T Q(x) V \succeq 0\}$.
5. The "complex spectrahedra" are spectrahedra: Let $\mathcal{H}_n$ denote the complex Hermitian space, and $Q : \Re^m \to \mathcal{H}_n$ be affine. We can split $Q(x) = R(x) + iC(x)$, where $R(x)$ is an affine real symmetric matrix map, and $C(x)$ is an affine real skew-symmetric matrix map. Then $Q(x)$ is Hermitian PSD if and only if
$$\begin{bmatrix} R(x) & C(x) \\ -C(x) & R(x) \end{bmatrix} \geqslant 0$$
and hence we obtain a spectrahedron (in [17], a similar construction is given for spectrahedra defined over the quaternion PSD cone).
6. Convex quadratic inequalities give spectrahedra: Let $f(x) = x^T L^T L x + b^T x + c$. Then $f(x) \leqslant 0$ if and only if
$$\begin{bmatrix} -(b^T x + c) & x^T L^T \\ Lx & I \end{bmatrix} \succeq 0.$$
7. If there exists an $\bar{x}$ such that $Q(\bar{x}) \succ 0$, then $\mathrm{Int}(G) \neq \emptyset$: For x sufficiently close to $\bar{x}$, $Q(x) \succ 0$. The converse is not necessarily true. However, there exists a (polynomial time computable) nonsingular matrix X such that $X^T Q(x) X = \tilde{Q}(x) \oplus 0$, and $\mathrm{Int}(G) = \{x | \tilde{Q}(x) \succ 0\}$ (SS 2.4).
8. If $Q(x)$ is linear, i.e. $Q_0 = 0$, then G is a convex cone. Again, the converse need not hold in general. Interestingly enough, the problem of checking whether a quartic function is convex can be reduced to determining whether a spectrahedron given by a matrix map is conical (SS 2.5). The complexity of the former problem is open, and its resolution is considered to be challenging [23].

1.5. Brief Summary of the Main Results

In Section 2, we investigate the structure of the faces of spectrahedra. We derive for any given $x \in G$, an expression for the face of $F_G(x)$ in terms of the null space map $\mathrm{Null}(Q(x))$. This yields a characterization of the faces of a spectrahedron. Also, we obtain the facts that the null space is constant over the relative interiors of the faces, and that the faces of spectrahedra are always exposed. We specialize the results to obtain characterizations for extreme points and extreme rays of spectrahedra. The two issues of when a spectrahedron is full-dimensional or conical are also addressed.

Certain results concerning the polars (denoted by G°) of spectrahedra are presented in Section 3. We define the "algebraic polar" G^* of a spectrahedron, and show that $G^\circ = \mathrm{Cl}(G^*)$. It is then shown that $G^\circ = G^* + (\mathrm{Aff}(G))^\perp$. Then we show that some polyhedral properties such as being closed under polar-taking, do not extend to spectrahedra.

2. Faces of Spectrahedra

Let us first recap the definition of the face of a convex set. A *face* of a convex set G is a convex subset F such that whenever $a, b \in G$ and $ta + (1-t)b \in F$ for some $0 < t < 1$, we have $a, b \in F$. As already mentioned, every point x of a convex set G belongs to the relative interior of a unique face, denoted by $F_G(x)$. Notice that

$$\begin{aligned}\mathrm{Aff}(F_G(x)) &= \{x + z | \exists t > 0 \text{ s.t. } x - tz, x + tz \in G\}\\ F_G(x) &= \mathrm{Aff}(F_G(x)) \cap G\end{aligned}$$

Henceforth, assume that G is a spectrahedron given by:

$$G = \{x | Q(x) \succeq 0\}.$$

2.1. A Characterization of Faces

In this subsection, we will give an algebraic description of the faces of G. This will then imply a characterization of faces. It will also follow that every face of a spectrahedron is exposed. We essentially generalize the following result for the faces of the PSD cone (a spectrahedron) which appears in [3].

LEMMA 1. *If $W \in \mathcal{P}_n$, then*

$$F_{\mathcal{P}_n}(W) = \{U \succeq 0 | \mathit{Null}(U) \supset \mathit{Null}(W)\}.$$

For any $x \in \Re^m$, let $\hat{Q}(x) = Q(x) - Q_0$ and $N(x) = \mathrm{Null}(Q(x))$.

We first prove the following technical lemma. Given $A, B \in \mathcal{S}_n$, and a subspace $S \subset \Re^n$ we say that "$A \succeq B$ over S", if $u^T(A - B)u \geqslant 0\ \forall u \in S$.

LEMMA 2. *Let $A, B \in \mathcal{S}_n$, $A \succeq 0$. Then*

$$A \succeq B \succeq -A$$

iff

(a) $A \succeq B \succeq -A$ *over* $\mathit{Null}(A)^\perp$ *and*

(b) *Null*$(B) \supset$ *Null*(A).

Proof. Put $N = \text{Null}(A)$.

Sufficiency. We need to show that

$$w^T Aw \geqslant w^T Bw \geqslant -w^T Aw \ \ \forall w \in \Re^n. \tag{1}$$

Given any $w \in \Re^n$, one can write $w = u + v$ for some $u \in N^\perp$ and $v \in N$. Then $w^T Aw = u^T Au$, and by (b), $w^T Bw = u^T Bu$. This together with (a) proves (1).

Necessity. Suppose that (1) holds. We may put $w = tu + v$ for some arbitrary $v \in N$, $u \in \Re^n$ and $t > 0$ to get

$$\begin{aligned} t^2u^T Au &\geqslant t^2u^T Bu + 2tu^T Bv + v^T Bv \\ &\geqslant -t^2u^T Au \ \ \forall u \in \Re^n, \quad v \in N, t > 0. \end{aligned} \tag{2}$$

Taking the limit as t goes to 0 in (2), we get $v^T Bv = 0 \, \forall v \in N$. Plugging this into (2) and dividing by t yields

$$tu^T Au \geqslant tu^T Bu + 2u^T Bv \geqslant -tu^T Au.$$

Once again taking the limit as t goes to 0, we get

$$u^T Bv = 0 \ \ \forall u \in \Re^n v \in N.$$

This implies that $Bv = 0$ for all $v \in N$. Thus (b) holds. Since (a) holds trivially, the proof is complete. □

THEOREM 1. *Let $\bar{x} \in G$ and define the affine subspaces:*

$$\begin{aligned} S_1(\bar{x}) &= \{z | N(z) \supset N(\bar{x})\} \\ S_2(\bar{x}) &= \{z | v^T Q(z) v = 0 \ \ \forall v \in N(\bar{x})\}. \end{aligned}$$

Then the following hold:

1. $F_G(\bar{x}) = S_1(\bar{x}) \cap G = S_2(\bar{x}) \cap G$

2. *Aff*$(F_G(\bar{x})) = S_1(\bar{x})$.

Proof. That $S_1(\bar{x}) \cap G = S_2(\bar{x}) \cap G$ is easily verified. The theorem will follow if we show the second assertion. For any nonzero $y \in \Re^m$, we have:

$$\begin{aligned} \bar{x} + y \in \text{Aff}(F_G(\bar{x})) &\Leftrightarrow \exists t > 0 \text{ such that } \bar{x} - ty, \bar{x} + ty \in G \\ &\Leftrightarrow \exists t > 0 \text{ such that } Q(\bar{x}) \succeq t\hat{Q}(y) \succeq -Q(\bar{x}) \\ &\Leftrightarrow \exists t > 0 \text{ such that} \\ &\qquad \text{(a) } Q(\bar{x}) \succeq t\hat{Q}(y) \succeq -Q(\bar{x}) \text{ over } N(\bar{x})^\perp \\ &\qquad \text{(b) } \text{Null}(\hat{Q}(y)) \supset N(\bar{x}). \end{aligned}$$

The second bi-implication comes from an application of Lemma 2. We claim that (a) is redundant in the above; given a y' satisfying (b), we will find a $t' > 0$

such that (a) is satisfied by $y = t'y'$: If $\hat{Q}(y') = 0$, take $y = y'$ and (a) holds since $Q(\bar{x}) \succeq 0$. Otherwise, (b) implies that $Q(\bar{x}) \neq 0$, and we choose t' to be λ/ρ, where λ is the smallest nonzero eigenvalue of $Q(\bar{x})$ and ρ is the spectral radius of $\hat{Q}(y)$. Then, for any unit-vector $v \in N^{\perp}$, $v^T Q(\bar{x}) v \geqslant \lambda$, and $|v^T \hat{Q}(y) v| \leqslant \rho$, and hence the stated claim follows. As a result, $z \in \text{Aff}(F_G(\bar{x}))$ iff $\text{Null}(\hat{Q}(z - \bar{x})) \supset N(\bar{x})$ iff $N(z) \supset N(\bar{x})$, proving the theorem. □

The corollary below is an easy consequence of the theorem.

COROLLARY 1. *Let G be a spectrahedron. Then*

1. *the null space $N(x)$ is constant over the relative interior of any face of G.*
2. *every face of G is exposed.*

Proof. Suppose that $z \in ri(F_G(\bar{x}))$. Then $F_G(z) = F_G(\bar{x})$, and hence $z \in S_1(\bar{x})$, implying that $N(z) \supset N(\bar{x})$. Similarly $N(\bar{x}) \supset N(z)$, proving the first assertion.

Let us now prove the second assertion assuming that $\bar{x} = 0$ and $Q(x)$ is in reduced form, i.e.

$$Q(x) = \begin{bmatrix} D + A(x) & B(x)^T \\ B(x) & C(x) \end{bmatrix},$$

where $C(x)$ is a $k \times k$ (linear) matrix map, and $D \succ 0$ ($A(x)$ and $B(x)$ are linear as well). If $k = 0$, then 0 is an interior point of G, so $F_G(0) = G$, and hence G is exposed. Assume that $k > 0$. By an application of the theorem, we obtain

$$S := \text{Aff}(F_G(0)) = \{x | C(x) = 0, B(x) = 0\}.$$

Define $a_i = tr(C_i)\ \forall i = 1, \ldots, m$. We claim that

$$F_G(0) = \{x \in G | a^T x = 0\}.$$

First, suppose that $x \in G$ and $a^T x = 0$. Since $C(x) \succeq 0$ and $a^T x = tr(C(x)) = 0$, we have that $C(x) = 0$ and $B(x) = 0$. Hence $x \in F_G(0)$. The reverse inclusion is easily shown. Therefore, if $a \neq 0$, then $F_G(0)$ is an exposed face. Suppose now that $a = 0$, implying that, for any $x \in G$, $C(x) \succeq 0$, $tr(C(x)) = 0$, and thus $C(x) = 0$. We conclude that $C(x) = 0$ for all $x \in G$. Consequently, $B(x) = 0$ for all $x \in G$, implying that $F_G(0)$ is G itself.

The proof of the general case follows after an application of a translation that sends $\bar{x}$ to the origin followed by a congruence transformation that sends the matrix map to the desired form. □

The fact that all the faces of spectrahedron are exposed will be used in Section 2.5 to show that the polars of spectrahedral cones need not be spectrahedral. The following is a simple characterization of the faces of G.

COROLLARY 2. *The faces of G are the maximal convex subsets S such that $N(x)$ is constant over $ri(G)$.*

2.2. Extreme Points and Rays

Theorem 1 will now be applied to obtain characterizations for the extreme points and extreme rays of spectrahedra. Recall that a point $\bar{x}$ is said to be an extreme point if there do not exist x, y distinct from $\bar{x}$ such that $\bar{x} = (x+y)/2$, or equivalently, $F_G(\bar{x}) = \{\bar{x}\}$.

COROLLARY 3. *Let G be a spectrahedron as defined above, and suppose that $\bar{x} \in G$. Then the following are equivalent:*

1. *$\bar{x}$ is an extreme point of G.*
2. $\forall y \in \Re^m$, *Null*$(\hat{Q}(y)) \supset N(\bar{x}) \Rightarrow y = 0$.
3. $z \in \Re^m$, $N(z) \supset N(\bar{x}) \Rightarrow z = \bar{x}$.
4. $z \in G$, $N(z) \supset N(\bar{x}) \Rightarrow z = \bar{x}$.

These last two conditions can be worded as: A point in a spectrahedron is extremal if and only if it is maximal w.r.t. the partial order of containment of the null spaces of the matrix map (maximality over G or equivalently over the whole space $\Re^m$).

Carrying out a similar exercise yields a characterization for the extreme rays of spectrahedral cones. We assume that $Q_0 = 0$. A nonzero vector x in a cone G is said to be irreversible, if $-x \notin G$, i.e., x is not in the lineal hull of G. An irreversible vector is said to be extremal if $\dim(\mathrm{Aff}(F_G(\bar{x}))) = 1$.

COROLLARY 4. *Let $Q_0 = 0$, and let $\bar{x}$ be an irreversible non-zero vector in the spectrahedral cone G. Then the following are equivalent:*

1. *$\bar{x}$ is extremal, or equivalently,* $\dim(\mathit{Aff}(F_G(\bar{x}))) = 1$.
2. *Null*$(Q(y)) \supset$ *Null*$(Q(\bar{x})) \Rightarrow y \in$ *span*$\{\bar{x}\}$.

It is easy to see that the dimensions of the faces of a polyhedron form a contiguous string of integers. This does not hold for spectrahedra: It is well known and follows easily from Lemma 1 that the faces of $\mathcal{P}_n$ have dimensions $k(k+1)/2$ for $k = 0, \ldots, n$, and hence there are "missing" dimensions in between these (triangular) integers.

2.3. Computational Aspects

Using Theorem 1, one can easily compute a basis for $\mathrm{Aff}(F_G(\bar{x}))$ for any given point $\bar{x} \in G$. To see this, suppose that u_i, $i = 1, \ldots, k$, span $N(\bar{x}) = \mathrm{Null}(Q(\bar{x}))$ (such a collection can be obtained by performing a Cholesky decomposition of $Q(\bar{x})$). Then a basis for the linear subspace

$$S = \{y | \hat{Q}(y)u_j = 0 \;\; \forall i = 1, \ldots, k\},$$

can be computed by Gaussian elimination. Then $\mathrm{Aff}(F_G(\bar{x})) = \bar{x} + S$.

Now, suppose that we want to check if $\bar{x}$ is an extreme point of G. Then, define the kn^m matrix

$$B = \begin{bmatrix} Q_1u_1 & \cdots & Q_mu_1 \\ \vdots & \ddots & \vdots \\ Q_1u_k & \cdots & Q_mu_k \end{bmatrix}.$$

By Corollary 3, $\bar{x}$ is an extreme point if and only if B has full column rank. If the matrices Q_i, $i = 0, \ldots, m$ as well as the vector $\bar{x}$ are rational vector, then all of the above computations are polynomial time.

As witnessed in this section, essentially, the null space $N(\bar{x})$ takes on the role of "the index set of active constraints" for the polyhedral case, which becomes a basis at a nondegenerate extreme point. This analogy can perhaps be used to develop simplex-like algorithms for Semidefinite Programming. Such an algorithm might "jump" from one $N(\bar{x})$ to another much like changing bases in linear programming.

2.4. Full-dimensional Spectrahedra

As mentioned in the introduction, if there exists an x such that $Q(x) \succ 0$, then $\mathrm{Int}(G)$ is not empty, but the converse is not necessarily true. However, we can "treat" the matrix in a straightforward manner, so that there is an exact correspondence.

Let us define the subspace

$$N = \bigcap_{i=0}^{m} \mathrm{Null}(Q_i),$$

and let V be a nonsingular matrix whose first k columns span N. Note that such a V can be computed in polynomial time. We then obtain

$$V^T Q(x) V = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{Q}(x) \end{bmatrix}.$$

Therefore, $G = \{x | \tilde{Q}(x) \succeq 0\}$. We now claim that $\mathrm{Int}(G)$ is nonempty if and only if there exists an x such that $\tilde{Q}(x) \succ 0$: Suppose that $\mathrm{Int}(G) \neq \emptyset$. Let N' be the constant null space (Corollary 1) over the interior. From the block diagonal form, it is clear that $N' \supset N$. Also,

$$\forall u \in N' \;\; \forall x \in \mathrm{Int}(G), \quad Q(x)u = 0.$$

Since a linear function vanishes on an open set iff it is identically zero, we conclude that $N' \subset N$, and hence $N' = N$. Thus, there exists a point x such that $\tilde{Q}(x) \succ 0$. Note that the above is a polynomial time process for obtaining $\tilde{Q}(x)$ whenever the matrix map $Q(x)$ has rational coefficients (same applies for arbitrary matrix maps in the real number model of computation). Hence we have, COROLLARY 5. *Let* $G = \{x|Q(x) \succeq 0\}$ *and let* N *be the intersection of the null spaces of* Q_i, $i = 0, \ldots, m$. *If* V *is a full rank matrix whose columns span the orthogonal complement of* N, *then*

$$G = \{x|V^T Q(x)V \succeq 0\}, \text{ and Int } (G) = \{x|V^T Q(x)V \succ 0\}.$$

2.5. Conical Spectrahedra

In contrast with the above, where full-dimensionality and its "natural" algebraic analog (i.e., that of having an x such that $Q(x) \succ 0$) are sufficiently close, the same is not the case with the conicity of spectrahedra as shown below.

A convex set is conical if $tx \in G$ for every $x \in G$ and $t \geqslant 0$. As noted before, if $Q_0 = 0$, then G is conical, and the converse need not hold (take $Q(z) = \text{Diag}(1 + x_1, x_1)$, for instance). As before, $\hat{Q}(x)$ will denote the pure-linear part of $Q(x)$. We will start by giving simple expressions for the recession cones and lineal hulls of spectrahedra.

Since $G = \{x|u^T \hat{Q}(x)u \geqslant -u^T Q_0 u \;\; \forall u \in \mathfrak{R}^n\}$, by a result in ([30], p. 62),

$$0^+(G) = \{x|u^T \hat{Q}(x)u \geqslant 0 \;\; \forall u \in \mathfrak{R}^n\}.$$

LEMMA 3. *The recession cone of* G *is given by* $\{x|\hat{Q}(x) \succeq 0\}$. *The lineality space of* G *is* $\{x|\hat{Q}(x) = 0\}$.

Since a closed convex set G is conical if and only if $0^+(G) = G$, a ready application of the lemma is the following simple characterization of conical spectrahedra.

COROLLARY 6. *The spectrahedron* $G = \{x|Q(x) \succeq 0\}$ *is conical iff* $G = \{x|\hat{Q}(x) \succeq 0\}$.

Now consider a matrix map in the following special form:

$$Q(x) = \begin{bmatrix} I + A(x) & 0 \\ 0 & C(x) \end{bmatrix},$$

where $A(x)$ and $C(x)$ are linear matrix maps, and define the cones

$$K_1 = \{x|A(x) \succeq 0\}, \text{ and } K_2 = \{x|C(x) \succeq 0\}.$$

COROLLARY 7. *If* $Q(x)$ *is of the above form, then* G *is conical iff* $K_1 \supset K_2$ *(iff* $G = K_2$*).*

Proof. If $K_1 \supset K_2$, then $G = K_2$, and hence it is conical. Conversely, suppose that G is a cone. Then by the above corollary, $G = K_1 \cap K_2$. Suppose that there exists an $x \in K_2$ such that $A(x) \not\succeq 0$. For sufficiently small $t > 0$, $I + A(tx) \succ 0$, implying $tx \in G$. But since G is assumed to be a cone, $t'x \in G$ for all $t > 0$. This is a contradiction since for sufficiently large t', $I+A(t'x) \not\succeq 0$. The lemma follows. □

The problem of checking if a given quartic polynomial is convex is an outstanding open problem in complexity theory [23]. In [27], it was shown that the convexity problem can be reduced to the problem of checking whether a certain spectrahedral cone (given by a linear matrix map) contains the PSD cone. From the corollary, this clearly implies that detecting whether a given spectrahedron is a cone is at least as hard as the quartic convexity problem.

3. Polars of Spectrahedra

The *(Geometric) Polar* of a convex set S is defined as follows:

$$S^\circ = \{y | y^T x \leqslant 1 \ \ \forall x \in S\}.$$

In this section, we will investigate the relationship between the geometric polar of a spectrahedron G given by a matrix map $Q(x)$, and its algebraic counterpart defined as follows. Let $\hat{Q}(x)$ be the linear part of $Q(x)$, and let $L : S_n \to \Re^m$ denote the adjoint of this map, i.e. $L(U)_i = U \cdot Q_i \ \forall i = 1, \ldots, m$. Then, the *Algebraic polar* of G (with respect to the representation $Q(x)$) is

$$G^*[Q(x)] = \{-L(U) | U \cdot Q_0 \leqslant 1, U \succeq 0\}.$$

(Simply denoted by G^* whenever confusion is not likely to arise.) We will begin by considering the following Primal-Dual pair of semidefinite programs [2].

$$\inf\{U \cdot Q_0 | U \cdot Q_i = -c_i \ \ \forall i, U \succeq 0\} \qquad \text{(Primal)}$$

$$\sup\{c^T x | Q(x) \succeq 0\} \qquad \text{(Dual)}$$

That $G^\circ \supset G^*$ follows from the fact that weak duality always holds for the above pair:

$$\begin{aligned} G^\circ &= \{y | \sup\{y^T x | Q(x) \succeq 0\} \leqslant 1\} \\ &\supset \{y | \inf\{U \cdot Q_0 | L(U) = -y, U \succeq 0\} \leqslant 1\} \\ &\supset \{-L(U) | U \cdot Q_0 \leqslant 1, U \succeq 0\} \\ &= G^*. \end{aligned}$$

The following simple example shows that equality need not hold between these two sets.

EXAMPLE. For the map

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & 0 \end{bmatrix},$$

we have the following:

$$\begin{aligned} G &= \Re_+ \times \{0\} \\ G^\circ &= (-\Re_+) \times \Re \\ G^* &= \{(0,0)\} \cup \{(x_1,x_2) | x_1 < 0, x_2 \in \Re\}. \end{aligned}$$

Thus $G^\circ \neq G^*$. But, $G^\circ = \mathrm{Cl}(G^*)$, and this is always the case as shown in the following lemma.

LEMMA 4. *Suppose that* $0 \in G$ *(i.e.* $Q_0 \succeq 0$*). Then* $G^\circ = Cl(G^*)$.

Proof. As already shown, $G^* \subset G^\circ$. Since G° is always closed, $\mathrm{Cl}(G^*) \subset G^\circ$.

To show the reverse inclusion, put $H = \mathrm{Cl}(G^*)$ and consider any $w \in H^\circ$. Then, we have

$$w^T(-L(U)) \leqslant 1 \text{ whenever } U \succeq 0, U \cdot Q_0 \leqslant 1. \qquad (*)$$

We claim that $Q(w) \succeq 0$. First, if $Q_0 = 0$, then (*) implies

$$Q(w) \cdot U \geqslant -1 \ \ \forall U \succeq 0,$$

which happens if and only if $Q(w) \succeq 0$.

Suppose that $Q_0 \neq 0$. Let $V \succ 0$, and choose $\lambda > 0$ such that $\lambda V \cdot Q_0 = 1$. Then

$$(\lambda V) \cdot Q(w) = (\lambda V) \cdot Q_0 + W^T L(\lambda V) = 1 + \lambda w^T L(V) \geqslant 0 \text{ by } (*).$$

Therefore, $Q(w) \cdot V \geqslant 0 \, \forall V \succ 0$, implying that $w \in G$, and hence $H^\circ \subset G$. But then,

$$H = H^{\circ\circ} \supset G^\circ,$$

and the proof is complete. □

From the lemma, the difference $G^\circ \backslash G^*$ is a set of measure zero. It is interesting to characterize this set of "missing" points. Our next result is a partial progress in this direction.

THEOREM 2. *Let* $G = \{x | Q(x) \succeq 0\}$ *with* $Q_0 \succeq 0$. *Then*

$$G^\circ = Cl(G^*) = G^* + Aff(G)^\perp$$

Proof. Put $S = \mathrm{Aff}(G)^\perp$, and let A be a $k \times m$ matrix with rank k such that $S = \mathrm{Range}(A)$. Since $v^T x = 0$ for any $v \in S$ and $x \in G$,

$$G^* \subset G^* + S \subset G^\circ = \mathrm{Cl}(G^*),$$

and hence it suffices to show that $G^* + S$ is closed, which is proved by induction on n. The base case of $n = 1$ is trivial.

Suppose that $-L(U(i)) + A^T\lambda(i) \to w$ for some $U(i) \succeq 0$, $U(i) \cdot Q_0 \leqslant 1$, and $\lambda(i) \in \Re^k$, $i = 1, \dots, \infty$. We need to show that there exist $U \succeq 0$ and λ such that $U \cdot Q_0 \leqslant 1$ and $w = -L(U) + A^T\lambda$. Suppose the contrary. We may assume that either $U(i)$ is unbounded or that $\lambda(i)$ is unbounded, for otherwise, any accumulation point of the joint sequence will be the desired $[U, \lambda]$. Now define the normalized sequence

$$[\hat{U}(i), \hat{\lambda}(i)] = [U(i), \lambda(i)]/(U(i) \cdot I + \|\lambda(i)\|) \ \forall i,$$

and assume, by passing to a subsequence if necessary, that the sequence converges to $(\hat{U}, \hat{\lambda})$. Note the following.

- At least one of $\hat{U}$ or $\hat{\lambda}$ is nonzero.
- $-L(U) + A^T\hat{\lambda} = \lim w/(U(i) \cdot I + \|\lambda(i)\|) = 0$.
- In fact, $\hat{U} \neq 0$. This can be seen as follows: if $\hat{U} = 0$, then $A^T\hat{\lambda} = 0$, which by the independence of the rows of A implies that $\hat{\lambda} = 0$, a contradiction.
- Also, since $0 \leqslant U(i) \cdot Q_0 \leqslant 1 \ \forall i$, it follows that $\hat{U} \cdot Q_0 = 0$.

Pick linearly independent vectors $x(j) \in G, \forall j = 1, \dots, m-k$, and so $Ax(j) = 0 \ \forall j$, implying that

$$x(j)^T L(\hat{U}) = x(j)^T(-L(\hat{U}) + A^T\hat{\lambda}) = 0 \ \ \forall j = 1, \dots, m-k$$

and hence $Q(x(j)) \cdot \hat{U} = 0 \ \forall j$. Now we employ the following fact (In [2], it is used to derive a complementary slackness result for SDP):

$$A, B \succeq 0, A \cdot B = 0 \Rightarrow AB = 0.$$

By applying the above, we get that $\hat{U}Q_0 = 0 = Q_0\hat{U}$, and also

$$Q(x(j))\hat{U} = \hat{U}Q(x(j)) = 0 \ \ \forall j = 1, \dots, m-k$$

which implies that

$$Q_l\hat{U} = \hat{U}Q_l = 0 \ \ \forall l = 1, \dots, m.$$

This is so, since otherwise, we will have a nonzero solution v to the system $v^T x(j) = 0 \ \forall j$, contradicting the independence of the $x(j)$.

Now, to complete the final part of the proof, we will first assume that

$$\hat{U} = I_{n'} \oplus O_{n-n'} \text{ for some } n' > 0.$$

It follows that the Q_l are of the form:

$$Q_l = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{Q}_l \end{bmatrix} \ \ \forall i.$$

By the induction hypothesis, lemma holds for $\tilde{Q}(x)$, and this is easily seen to imply the same for $Q(x)$. Since the nonzero PSD matrix $\hat{U}$ can always be brought in to the above form by an orthogonal transformation, the Theorem follows. □

We conjecture the following improvement of the theorem.

CONJECTURE 1.

$$G^\circ = G^* \cup \mathrm{Aff}(G)^\perp$$

The theorem implies that when G has nonempty interior, then $G^* = G^\circ$. It also shows that for any matrix map, there exists a "rectified" map (one choice being $Q(x) \oplus \mathrm{Diag}(Ax) \oplus \mathrm{Diag}(-Ax)$) that gives the same spectrahedron, but its algebraic and geometric polars coincide.

3.1. Comparisons with Polyhedra

Let us collect the following properties of polyhedra, and inquire whether they extend to spectrahedra.

1. Faces are exposed
2. Closed under intersections
3. Dimensions of the faces form a contiguous string
4. Closed under linear maps
5. Closed under projections and Minkowski sums
6. Closed under polar taking

As we have seen, properties 1 and 2 extend to spectrahedra, and 3 does not. To see that 4 fails for spectrahedra, let us return to the example given before Lemma 4, and note that $G^* = \{(U_{11}, 2U_{12}) | U \in \mathcal{P}_2\}$. Since G^* is not closed, it is not spectrahedral. Also, G^* is the projection of $\mathcal{P}_2$ onto the subspace $\{U \in \mathcal{S}_n | U_{22} = 0\}$, and hence projections need not preserve spectrahedrality. We now give an example of two bounded spectrahedra, whose sum is not spectrahedral. Consider the following spectrahedra in $\Re^2$:

$$C = \{x \mid \|x\|_\infty \leqslant 2\} \text{ and } B = \{x \mid \|x\|_2 \leqslant 1\}.$$

The sum $C + B$ is a square with rounded corners. Pick any of the eight points on this set where a circular boundary meets a straight line, say $(3, 2)$. It is easy to see that this point is an unexposed extreme point. Since the faces of spectrahedra must be exposed, it follows that $C + B$ is not a spectrahedron.

A slight modification of the above example shows that the polars of spectrahedral cones are not necessarily spectrahedral: Let C be the Ice Cream Cone in $\Re^3$, i.e. $C = \{x | x_3 \geqslant \sqrt{x_1^2 + x_2^2}\}$. Note that $C = \{x | Q(x) \succeq 0\}$, where

$$Q(x) = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \end{bmatrix}.$$

Let K denote the spectrahedral cone $-(C \cap \Re^3_+)$, and observe that $K^\circ = C + \Re^3_+$. Clearly,

$$C + \Re^3_+ = C \cup \{x | x_3 \geqslant 0, x_2 + x_3 \geqslant 0, x_1 + x_3 \geqslant 0, x_3 - x_1 - x_2 \geqslant 0\},$$

and the plane $x_2 + x_3$ is tangential to C at $(0, -1, 1)$. It is not difficult to show that the ray generated by this vector is an unexposed extreme ray, implying that K° is not spectrahedral.

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Note

[1] The arithmetic complexity of their overall algorithm is polynomial. Unfortunately, neither the authors establish an explicit polynomial bound for the bitlengths of the intermediate numbers, nor does such a bound follow trivially from other results in the literature.

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